Xalq Ta'limi vazirligi Fan olimpiadalari bo'yicha iqtidorli o'quvchilar bilan ishlash departamentining haftalik olimpiadasi topshiriqlarining yechimlari

10-11 sinf o'quvchilari uchun

1. Aytaylik $A=1+\cos\frac{4\pi}{19}+\cos\frac{6\pi}{19}-\cos\frac{9\pi}{19}$, $B=1+\cos\frac{2\pi}{19}-\cos\frac{3\pi}{19}-\cos\frac{5\pi}{19}$ va $C=1+\cos\frac{\pi}{19}-\cos\frac{7\pi}{19}+\cos\frac{8\pi}{19}$ bo'lsin. U holda quyidagi tenglikni isbotlang:

$$\left(\sqrt[3]{A}+\sqrt[3]{B}+\sqrt[3]{C}\right)^3=\frac{3\sqrt[3]{19}-1}{2}$$

Yechimi: $\cos x=\cos(\pi-x)$ ayniyatga ko'ra A, B, C ifodalarni quyidagicha yozib olamiz:

$$A=1-\left(\cos\frac{15\pi}{19}+\cos\frac{13\pi}{19}+\cos\frac{9\pi}{19}\right) \quad \textbf{(1)}$$

$$B=1-\left(\cos\frac{17\pi}{19}+\cos\frac{3\pi}{19}+\cos\frac{5\pi}{19}\right) \quad \textbf{(2)}$$

$$C=1-\left(\cos\frac{18\pi}{19}+\cos\frac{7\pi}{19}+\cos\frac{11\pi}{19}\right) \quad \textbf{(3)}$$

Quyidagi belgilashlarni olaylik:

$$x_1=\cos\frac{15\pi}{19}+\cos\frac{13\pi}{19}+\cos\frac{9\pi}{19} \quad \textbf{(4)}$$

$$x_2=\cos\frac{17\pi}{19}+\cos\frac{3\pi}{19}+\cos\frac{5\pi}{19} \quad \textbf{(5)}$$

$$x_3=\cos\frac{18\pi}{19}+\cos\frac{7\pi}{19}+\cos\frac{11\pi}{19} \quad \textbf{(6)}$$

U holda x_1, x_2, x_3 sonlar

$$x^3-x^2-6x+7=0 \quad \textbf{(7)}$$

tenglamaning ildizlari ekanligini tekshirib ko‘rishimiz mumkin. $i=1,2,3$ indekslar uchun $y_i=2-x_i$ almashtirish bajarsak,

$$x^3-x^2-6x+7=0\Rightarrow y^3-5y^2+2y+1=0 \qquad \textbf{(7')}$$

munosabatni topamiz. Qulaylik uchun

$$\sqrt[3]{y_1}=a,\ \sqrt[3]{y_2}=b,\ \sqrt[3]{y_3}=c \qquad \textbf{(8)}$$

belgilashni kiritaylik. U holda masala

$$a+b+c=\sqrt[3]{3\cdot\sqrt[3]{19}+1} \qquad \textbf{(9)}$$

tenglikni isbotlashga teng kuchli. Viyet teoremasiga ko‘ra quyidagi tenglikni topamiz:

$$\begin{cases} a^3+b^3+c^3=y_1+y_2+y_3=5 \\ a^3b^3+b^3c^3+c^3a^3=y_1y_2+y_2y_3+y_3y_1=2 \\ a^3b^3c^3=y_1y_2y_3=-1 \end{cases} \qquad \textbf{(10)}$$

Quyidagi yordamchi parametrlarni belgilaymiz:

$$\begin{cases} p=a+b+c \\ q=ab+bc+ca \\ r=abc=-1 \end{cases} \qquad \textbf{(11)}$$

U holda

$$\begin{cases} p^3-3pq-3=p(p^2-3q)+3abc=a^3+b^3+c^3=5 \\ q^3+3pq+3=q(q^2-3rp)+3r^2=a^3b^3+b^3c^3+c^3a^3=2 \\ r=-1 \end{cases} \Rightarrow \begin{cases} p^3-3pq=8 \\ q^3+3pq=-1 \\ r=-1 \end{cases} \Rightarrow \qquad \textbf{(12)}$$

$$\begin{cases} p^3+q^3=7 \\ (p^3-8)^3=27p^3q^3=27p^3(7-p^3) \end{cases} \Rightarrow (p^3+1)^3=513=27\cdot 19 \Rightarrow p=\sqrt[3]{3\cdot\sqrt[3]{19}-1} \qquad \textbf{(13)}$$

Masala to‘liq yechildi.

2. $\{1,2,3,4,5,6,7,8,9,10,11\}$ to‘plamning shunday qism to‘plamlari sonini topingki, unda ketma-ket kelgan ikkita son yotmasin.

Yechimi: Masalani umumiy holatda yechamiz. Ketma-ket kelgan sonni o‘z ichiga olmaydigan to‘plamni ***yaxshi*** to‘plam deylik. $A_n=\{1,2,...,n\}$ to‘plamning ***yaxshi*** qism to‘plamlari sonini $f(n)$ bilan belgilaylik. Biz $A_{n+2}=\{1,2,...,n+2\}$ to’plam uchun ***yaxshi*** qism to‘plamlar sonini, ya’ni $f(n+2)$ ni hisoblaymiz. A_{n+2} to‘plamning $n+2$ ni o‘z ichiga olmagan ***yaxshi*** qism to‘plamlari $A_{n+1}=\{1,2,...,n+1\}$ to‘plamning ham yaxshi qism to‘plami bo‘ladi, ya’ni ular $f(n+1)$ ta. Boshqa tomondan A_{n+2} to‘plamning $n+2$ ni o‘z ichiga olgan ***yaxshi*** qism to‘plamlari $n+1$ ni o‘z ichiga olmaydi, ya’ni $n+2$ ni hisobga olmaganda $A_n=\{1,2,...,n\}$ to‘plamning yaxshi qism to‘plamlari bo‘ladi, ya’ni ularning soni $f(n)$ ta. Demak $f(n+2)=f(n+1)+f(n)$ rekkurent formula o‘rinli. Endi boshlang‘ich qiymatlarini hisoblaymiz:

$A_1=\{1\}$ to‘plamning yaxshi qism to‘plamlari: $\{1\}$ va $\varnothing \Rightarrow f(1)=2$ **(14)**

$A_2=\{1,2\}$ to‘plamning yaxshi qism to‘plamlari: $\{1\},\{2\}$ va $\varnothing \Rightarrow f(2)=3$ **(15)**

U holda $f(3)=2+3=5,\ f(4)=8,\ f(5)=13,\ .....\ f(11)=233$.

Javob: $f(11)=233$

3. $y^3-2x^2=5$ tenglamaning barcha butun yechimlarini toping.

Yechimi: Natural sonning kvadratini 8 ga bo‘lgandagi qoldirlarni tekshirish orqali

$$y^3=2x^2+5\in\{5,7\}\ (\text{mod } 8) \quad \textbf{(16)}$$

munosabatni topamiz. Demak $y\in\{5,7\}\ (\text{mod } 8)$ munosabat o‘rinli. Aytaylik $y\equiv 5\ (\text{mod } 8)$ bo‘lsin. Bu holda

$$y^2+y+1\equiv 7\ (\text{mod } 8) \quad \textbf{(17)}$$

bo‘lib, u $p\in\{5,7\}\ (\text{mod } 8)$ ko‘rinishidagi tub bo‘luvchiga ega. U holda

$$(y-1)(y^2+y+1)=2(x^2+2) \quad \textbf{(18)}$$

Tenglikka ko‘ra, x^2+2 ham $p\in\{5,7\}\ (\text{mod } 8)$ ga bo‘linadi, ya’ni

$$\left(\frac{-2}{p}\right)=1 \quad \textbf{(19)}$$

bo‘lishi kerak. Lekin oxirgi tenglik faqatgina $p\in\{1,3\}\ (\text{mod } 8)$ holdagina bajariladi. $y\equiv 7\ (\text{mod } 8)$ bo‘lgan holni o‘quvchilar mustaqil yechishlari uchun qoldiramiz.

4. $(1+x+x^2+x^3+x^4+x^5)^{402}=a_{2010}x^{2010}+a_{2009}x^{2009}+...+a_1x+a_0$ ko‘phadni qaraylik. U holda quyidagi nisbatni hisoblang:

$$\frac{a_1+2a_2+...+2009a_{2009}+2010a_{2010}}{a_0+a_1+a_2+...+a_{2010}}$$

Yechimi: Masalada berilgan ko‘phadni quyidagicha belgilaylik:

$$P(x) = (1 + x + x^2 + x^3 + x^4 + x^5)^{402} = a_{2010}x^{2010} + a_{2009}x^{2009} + ... + a_1 x + a_0 \quad \textbf{(20)}$$

Quyidagi tengliklar o‘rinli ekanligiga ishonch hosil qilish mumkin:

$$P'(x) = 2010a_{2010}x^{2009} + 2009a_{2009}x^{2008} + ... + 2a_2 + a_1 \Rightarrow$$

$$P'(1) = a_1 + 2a_2 + ... + 2009a_{2009} + 2010a_{2010} \quad \textbf{(21)}$$

$$P(1) = a_0 + a_1 + a_2 + ... + a_{2010} = (1+1+1+1+1+1)^6 = 6^6 \quad \textbf{(22)}$$

Hosilani va uning 1 dagi qiymatini hisoblaymiz:

$$P'(x) = 402(1 + x + x^2 + x^3 + x^4 + x^5)^{401} \cdot (5x^4 + 4x^3 + 3x^2 + 2x + 1) \Rightarrow$$

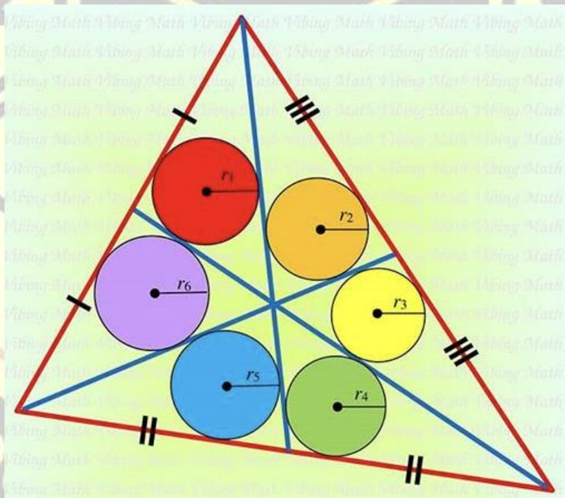
$$P'(1) = 402 \cdot 6^{401} \cdot 15 \quad \textbf{(23)}$$

Demak

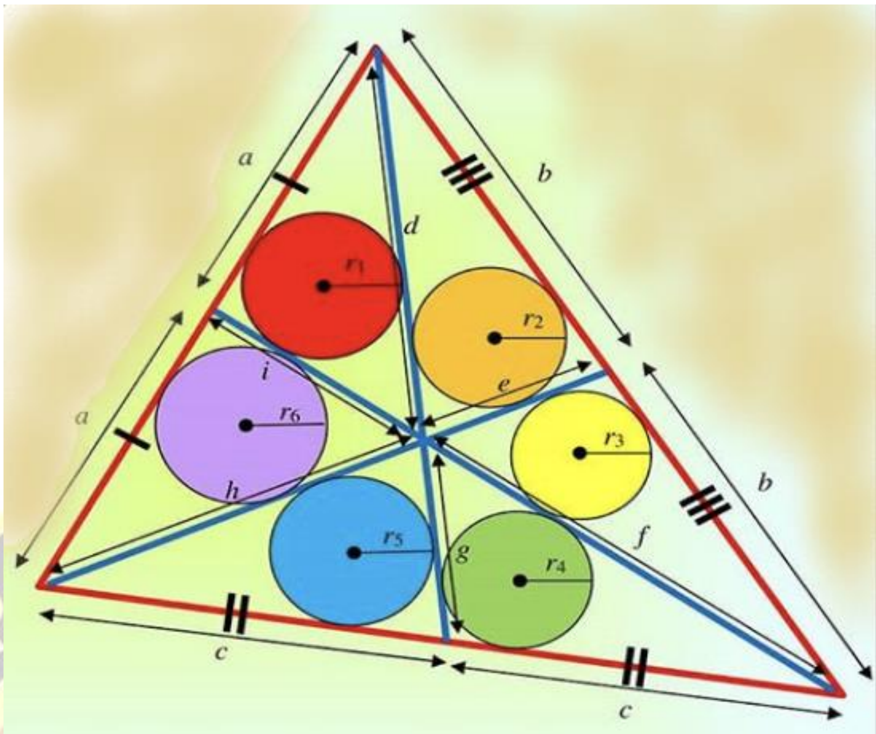
$$\frac{a_1 + 2a_2 + ... + 2009a_{2009} + 2010a_{2010}}{a_0 + a_1 + a_2 + ... + a_{2010}} = \frac{402 \cdot 6^{401} \cdot 15}{6^{402}} = 1005 \quad \textbf{(24)}$$

Javob: 1005

5. Quyidagi chizmaga asosan $\frac{1}{r_1} + \frac{1}{r_3} + \frac{1}{r_5} = \frac{1}{r_2} + \frac{1}{r_4} + \frac{1}{r_6}$ tenglikni isbotlang.



Yechimi: Kesmalarni quyidagi chizmadagidek belgilab olaylik.



Uchburchak medianasining xossasiga ko‘ra chizmadagi oltita uchburchak yuzalari bir xil. Ularni S bilan belgilaylik. Uchburchak yuzasi va ichki chizilgan aylanasi radiuslari orasidagi bog‘lanishga ko‘ra

$$\frac{1}{r_1}=\frac{a+d+i}{2S},\ \frac{1}{r_3}=\frac{b+e+f}{2S},\ \frac{1}{r_5}=\frac{c+g+h}{2S} \qquad \textbf{(25)}$$

va

$$\frac{1}{r_2}=\frac{b+d+e}{2S},\ \frac{1}{r_4}=\frac{f+c+g}{2S},\ \frac{1}{r_6}=\frac{a+h+i}{2S} \qquad \textbf{(26)}$$

tengliklarni topamiz. U holda

$$\frac{1}{r_1}+\frac{1}{r_3}+\frac{1}{r_5}=\frac{a+d+i+b+e+f+c+g+h}{2S} \qquad \textbf{(27)}$$

va

$$\frac{1}{r_2}+\frac{1}{r_4}+\frac{1}{r_6}=\frac{b+d+e+f+c+g+a+h+i}{2S} \qquad \textbf{(28)}$$

Demak

$$\frac{1}{r_1}+\frac{1}{r_3}+\frac{1}{r_5}=\frac{1}{r_2}+\frac{1}{r_4}+\frac{1}{r_6} \qquad \textbf{(29)}$$

tenglik o‘rinli ekan. Masala sharti to‘liq isbotlandi.

7-9 sinf o'quvchilari uchun

1. $x^3-1=\sqrt{x}(5x-3x^2-3)$ tenglamani yeching.

Yechimi: Tenglama berilishiga ko'ra $x>0$ va $\sqrt{x}=m$ kabi belgilaylik. U holda berilgan tenglama

$$m^6+3m^5-5m^3+3m-1=0 \quad \textbf{(30)}$$

ko'rinishiga keladi. $m\neq 0$ ekanligini inobatga olib, **(30)** tenglamaning har ikkala tarafini m^3 ga bo'lamiz:

$$m^3-\frac{1}{m^3}+3\left(m^2+\frac{1}{m^2}\right)-5=0 \quad \textbf{(31)}$$

Endi $m-\frac{1}{m}=n$ belgilash kiritib,

$$m^3-\frac{1}{m^3}=\left(m-\frac{1}{m}\right)^3+3\left(m-\frac{1}{m}\right)=n^3+3n \quad \textbf{(32)}$$

$$m^2+\frac{1}{m^2}=\left(m-\frac{1}{m}\right)^2+2=n^2+2 \quad \textbf{(33)}$$

ayniyatlardan foydalansak () tenglama quyidagi ko'rinishga keladi:

$$n^3+3n+3(n^2+2)-5=0 \quad \textbf{(34)}$$

yoki

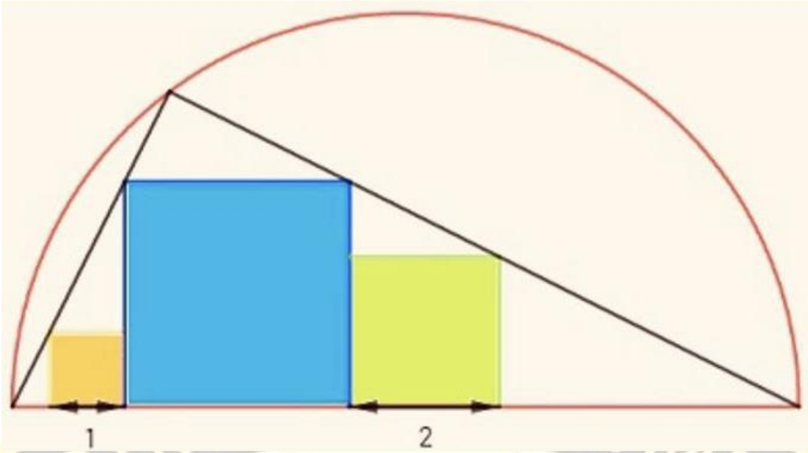
$$n^3+3n+3n^2+1=0 \Rightarrow (n+1)^3=0 \Rightarrow n=-1 \quad \textbf{(35)}$$

Demak

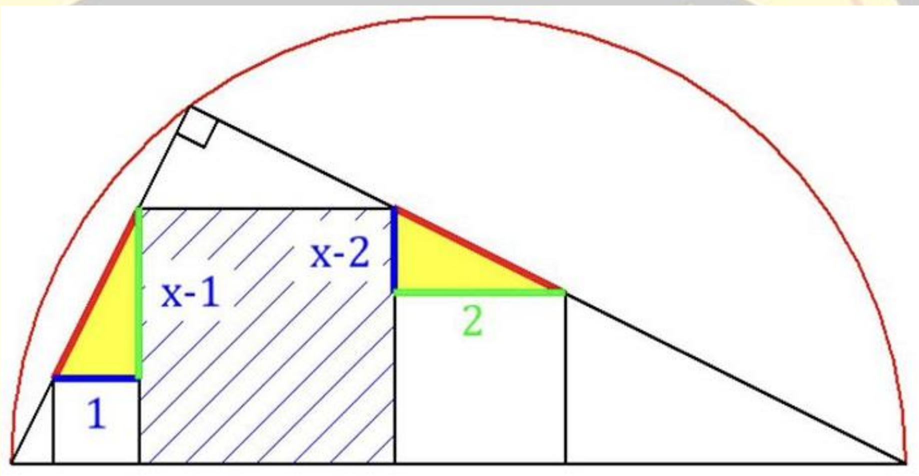
$$m-\frac{1}{m}=-1 \Rightarrow m^2+m-1=0 \Rightarrow \langle m>0\rangle \Rightarrow x=m^2=\left(\frac{\sqrt{5}-1}{2}\right)^2 \quad \textbf{(36)}$$

Javob: $x=\frac{3-\sqrt{5}}{2}$

2. Chizmadagi ma’lumotlarga asosan ko‘k rangli kvadrat yuzini toping.



Yechimi: Dastlab masalaga mos chizma chizib olaylik, bu yerda x -ko‘k rangli kvadratning tomoni uzunligi. U holda burchaklarni hisoblash orqali sariq rangli uchburchaklar o‘xshashligini aniqlash qiyin emas.



U holda o‘xshashlikka ko‘ra

$$\frac{1}{x-1}=\frac{x-2}{2}\Rightarrow(x-1)(x-2)=2\Rightarrow x(x-3)=0\Rightarrow\begin{cases}x=0\ \Rightarrow\ \varnothing\\x=3\end{cases}\qquad \textbf{(37)}$$

Demak kvadratning yuzasi 9 ga teng.

Javob: 9

3. Agar $\frac{1}{7}=0{,}142857...$bo’lsa, u holda verguldan keyingi 2020-xonadagi raqamni toping.

Yechimi: Faraz qilaylik 2020-xonadagi raqam x bo‘lsin. U holda

$$\frac{1}{7}=0{,}142857...x...\Rightarrow\frac{6}{7}=0{,}857142...\qquad \textbf{(38)}$$

$$10^6 \equiv 1 \ (\mathrm{mod}\ 7) \Rightarrow 10^{2019} \equiv 10^3 \equiv 6 \ (\mathrm{mod}\ 7) \qquad \textbf{(39)}$$

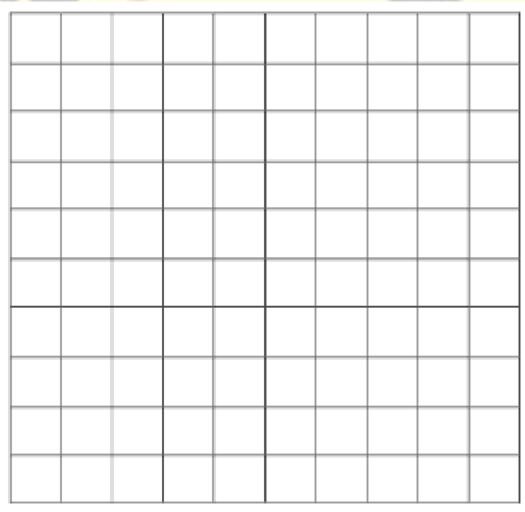
Demak

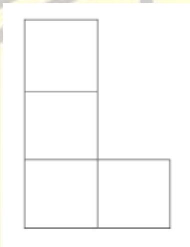
$$\frac{10^6-6}{7} \in \mathbb{Z} \Rightarrow 142857.....,x...-0,857142... \in \mathbb{Z} \Rightarrow \qquad \textbf{(40)}$$

$$0,x... = 0,857142... \Rightarrow x = 8 \qquad \textbf{(40΄)}$$

Javob: 8

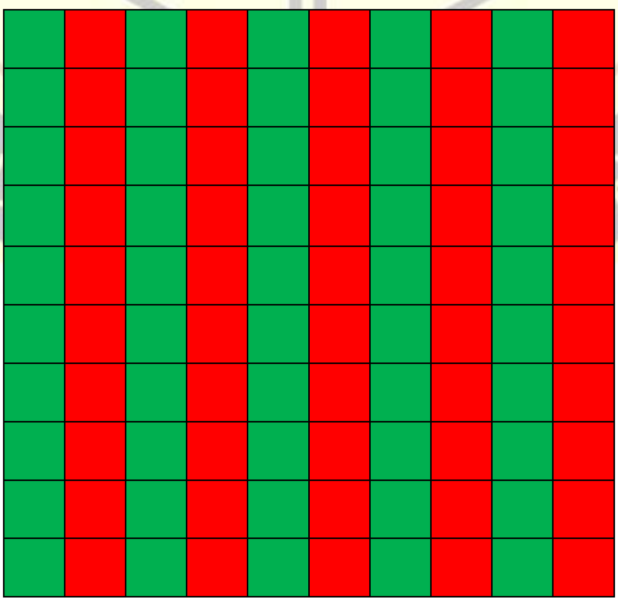
4. Quyidagi 100 ta katakdan iborat **A shaklni** 4 ta katakdan iborat **L** simon **B shakl**dagi 25 ta bo‘lakka bo‘lish mumkinmi? Bunda B shaklni burish mumkin, lekin ustma-ust qo‘yish mumkin emas. Javobingizni isbotlang.





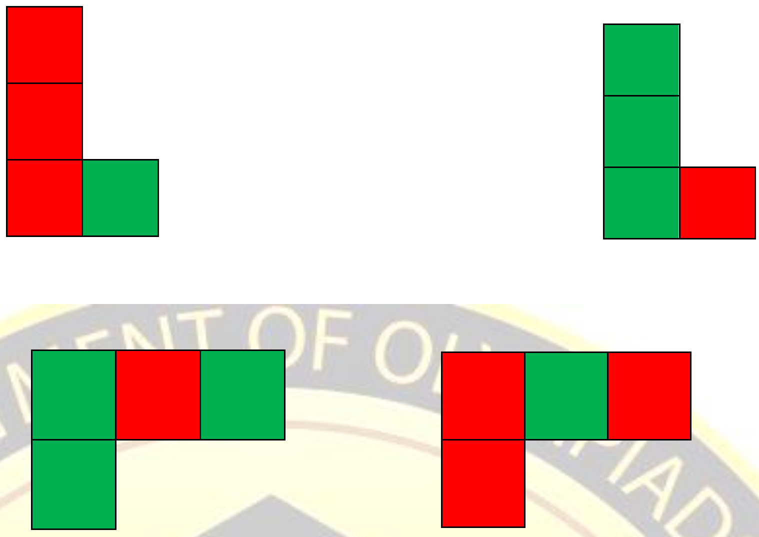
A shakl **B shakl**

Yechimi: Teskarisini faraz qilaylik, masala shartini bajarish mumkin bo‘lsin. **A shakl**ni quyidagi usulda yashil va qizil ranglarga bo‘yaylik.



A shakl

U holda **B shakl** qanday tarzda joylashishiga bog‘ilk quyidagi hollar bo‘lishi mumkin:



yoki ularning burilishlaridan iborat. Demak qanday holatda ham, 3 ta yashil va 1 ta qora yoki 3 qizil va 1 ta yashil ranglardan tashkil topishi mumkin ekan. 3 ta yashil rangli katakni o‘z ichiga olgan **B shakl**lar sonini g va 3 ta qizil rangli katakni o‘z ichiga olgan **B shakl**lar soni r bo‘lsin. U holda qizil va yashil rangli kataklar sonini hisoblaymiz:

$$\begin{cases} 3r+q=50 \\ 3q+r=50 \end{cases} \Rightarrow q=r=\frac{25}{2} \Rightarrow \varnothing \qquad \textbf{(41)}$$

ziddiyat. Demak farazimiz noto‘g‘ri.

5. Agar $a>b$ natural sonlar uchun $(a^2+1)(b^2+1)$ ko‘paytma biror natural sonning kvadratiga teng bo‘lsa, u holda $a>5b$ tengsizlikni isbotlang.

Yechimi: Quyidagi ifodani qaraylik:

$$c^2=(a^2+1)(b^2+1)=(ab+1)^2+(a-b)^2>(ab+1)^2 \qquad \textbf{(42)}$$

Demak

$$c^2=(a^2+1)(b^2+1)\geq(ab+2)^2 \qquad \textbf{(43)}$$

Tengsizlik o‘rinli. Faraz qilaylik tenglik holi bajarilsin, u holda

$$(a^2+1)(b^2+1)=(ab+2)^2 \Rightarrow a^2+b^2=4ab+3 \Rightarrow \varnothing \qquad \textbf{(44)}$$

Chunki, ikkita sonning kvadratlari yig‘indisini 4 ga bo’lganda 3 qoldiq qolmaydi. Demak

$$(a^2+1)(b^2+1)\geq(ab+3)^2 \Rightarrow a^2+b^2=6ab+8>6ab \Rightarrow$$

$$\left\langle \frac{a}{b}=x \right\rangle \Rightarrow x^2+1>6x \Rightarrow (x-3)^2>8 \Rightarrow x>3+2\sqrt{2}>5 \qquad \textbf{(45)}$$

Masala to'liq isbotlandi.

4-6 sinf o'quvchilari uchun

1. Quyidagi rebusni yeching:

$$\begin{array}{r} A\,A\,A\,A \\ +\quad A\,A\,A\,B \\ \hline C\;C\,C\,C\,C \end{array}.$$

Yechimi: Agar ikkita to'rt xonali son yig'indisi besh xonali bo'lsa, u holda ushbu besh xonali sonning birinchi raqami 1 ga teng bo'ladi. Demak $C=1$. U holda

$$\overline{AAAA}+\overline{AAAB}=11111 \Rightarrow 2221\cdot A+B=11111 \qquad \textbf{(46)}$$

Tekshirib ko'rish orqali $A=5$ va $B=6$ tengliklarni topamiz.

2. To'rtta 4 raqami va arifmetik amallardan foydalanib 0, 1, 2, … 19, 20 sonlarini hosil qiling. *(Masalan* $4!-4-4:4=19$*)*

Yechimi: Quyidagi hosil qilish usuli keltirilgan.

$(4+4)-(4+4)=0$

$(4+4):(4+4)=1$

$4:4+4:4=2$

$4-4^{4-4}=3$

$4+(4-4)\cdot 4=4$

$4+4^{4-4}=5$

$4+(4+4):4=6$

$4+4-4:4=7$

$4+4+4-4=8$

$4+4+4:4=9$

$(44-4):4=10$

$44:(\sqrt{4}\cdot\sqrt{4})=11$

$4\cdot(4-4:4)=12$

$44:4+\sqrt{4}=13$

$4+4+4+\sqrt{4}=14$

$44:4+4=15$

$4\cdot 4+4-4=16$

$4\cdot 4+4:4=17$

$4\cdot 4+4-\sqrt{4}=18$

$4!-4-4:4=19$

$$4\cdot 4+\sqrt{4}+\sqrt{4}=20$$

Albatta bu yagona yechim emas.

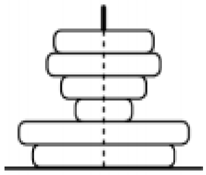
3. Nodir 2, 0, 1 va 8 raqamlari yordamida 2 xonali son tuzmoqda. Bu son 10 dan katta ammo 25 dan kichik bo'lishi kerak. Har bir son 2 ta turli raqamdan tuziladi. Nodir necha xil usulda shunday sonlarni tuzishi mumkin?

Yechimi: Masala shartini qanoatlantiruvchi sonlar:

12, 18, 20, 21 **(47)**

Javob: 4 ta

4. Dilshod quyidagi o‘yinchoq minorani disklar yordamida qurdi va bu minoraga tepadan qaradi. U nechta diskni ko‘radi?



Javob: 2 ta

5. Fermada 15 ta hayvon yashaydi: sigirlar, mushuklar va kengurular. 10 ta hayvon sigir emas va 8 ta hayvon mushuk emas. Fermadagi kengurular sonini toping.

Yechimi: Masala bir qarashda xatodek ko‘rinishi mumkin. Lekin diqqat bilan tushunib o‘qisangiz asl ma’nosiga tushunasiz. 10 ta hayvon sigir emas degani, mushuklar va kengular jami 10 ta degani, ya’ni 5 tasi sigir ekan. Xuddi shunday 8 ta hayvon mushuk emas degani, sigirlar va kengurular 8 ta degani, ya’ni mushuklar 15-8=7 ta degani. U holda kengurular soni 15-5-7=3 ta ekan.

Javob: 3 ta.

Fan olimpiadalari bo‘yicha iqtidorli o‘quvchilar bilan ishlash departamenti sizga omadlar tilaydi!