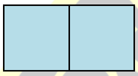
Xalq ta’limi vazirligi Fan olimpiadalari bo‘yicha iqtidorli o‘quvchilar bilan ishlash departamentining matematika fanidan haftalik topshiriqlari yechimlari

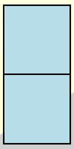
10-11 sinf o‘quvchilari uchun

1 – masala. Quyidagi 8×8 jadvaldan ikkita uchidagi kataklari olingan figurani qaraylik. Bu shaklni 31 ta 1×2 yoki 2×1 dominolarga bo‘lish mumkinmi?



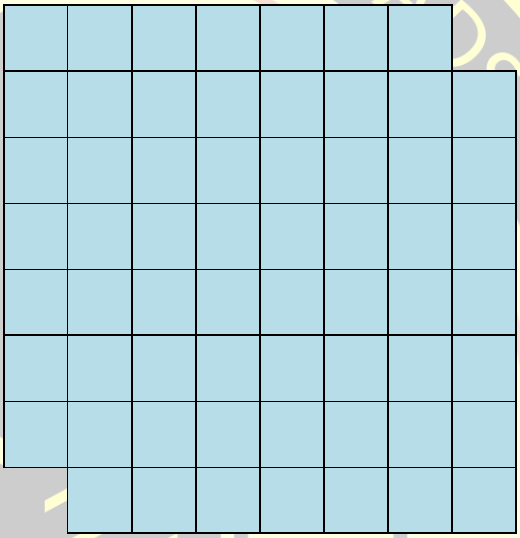
1×2

Domino

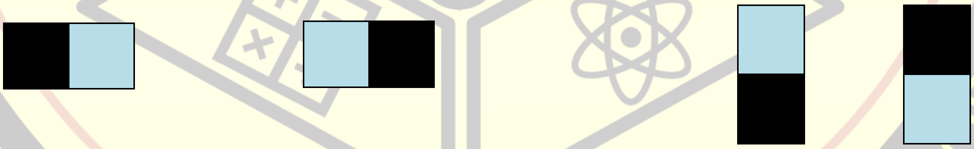


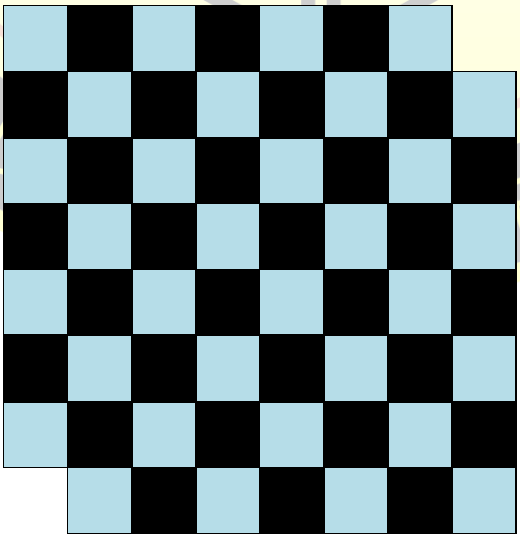
2×1

Domino



Yechimi: Quyidagicha bo‘yaylik.





Agar masalada talab qilinganidek dominolar bilan bo‘lish mumkin bo‘lsa, qora va ko‘k kataklar soni teng bo‘lishi kerak. Lekin qora katalar 30 ta ko‘k kataklar esa 32 ta. Demak dominolar bilan bo‘lish mumkin emas.

2 – masala. Ixtiyoriy qavariq $ABCD$ to‘rtburchak uchun quyidagi

$$(AC\cdot BD)^2=(AB\cdot CD)^2+(BC\cdot AD)^2-2AB\cdot BC\cdot CD\cdot DA\cos(\angle B+\angle D)$$

tenglikni isbotlang.

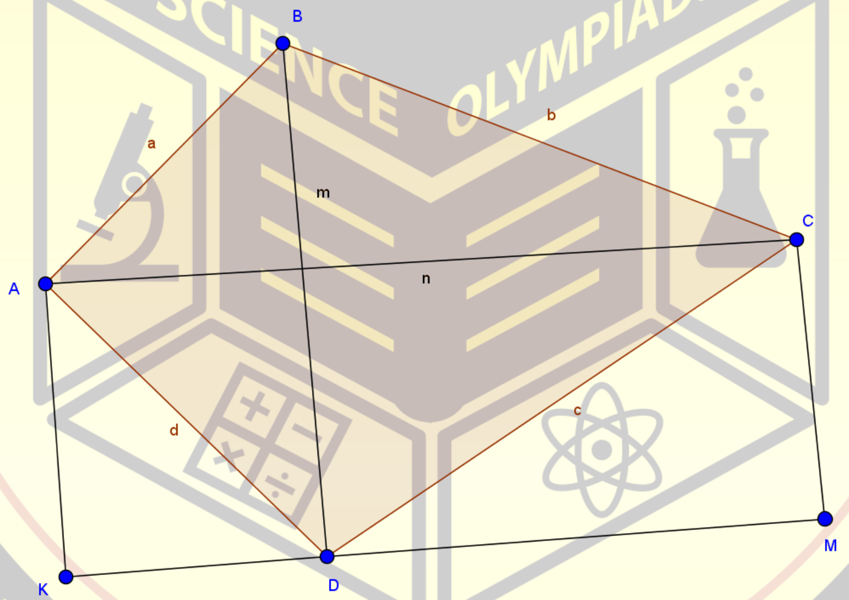
Yechimi: Qulaylik uchun

$$AB=a,\ BC=b,\ CD=c,\ AD=d,\ DB=m,\ AC=n$$

kabi belgilash olaylik. Berilgan qavariq to‘rtburchakning AD va DC tomonlariga tashqi tomondan quyidagi shartlarni qanoatlantiruvchi uchburchaklar yasaymiz:

$$\triangle AKD\sim\triangle DCB:\ \angle KAD=\angle BDC=z,\ \angle ADK=\angle DBC=x$$

$$\triangle DMC\sim\triangle BAD:\ \angle ADB=\angle DCM=y,\ \angle ABD=\angle CDM=t$$



O‘xshash uchburchaklardan quyidagi tengliklarni aniqlaymiz:

$$\triangle AKD\sim\triangle DCB\Rightarrow\frac{AK}{DC}=\frac{DA}{BD}\Rightarrow AK=\frac{AD}{BD}\cdot DC=\frac{dc}{m},\ KD=\frac{bd}{m}\ \textbf{(2)}$$

$$\triangle DMC\sim\triangle BAD\Rightarrow\frac{CM}{AD}=\frac{DC}{BD}\Rightarrow CM=\frac{dc}{m}\Rightarrow AK=CM\ \textbf{(3)}$$

Ravshanki

$$(y+z)+\angle DAC+\angle ACD=180^\circ\ \Rightarrow\ \angle KAC+\angle MCA=180^\circ$$

tenglik o‘rinli. Bundan $AK\,||\,CM$ munosabat kelib chiqadi. Boshqa tomondan **(3)** tenglikka ko‘ra $AK=CM$ ga egamiz. Bu topilgan munosabatlardan $KACM$ parallelogram ekanligi kelib chiqadi. U holda $AC = KM = n$. Shuningdek $\triangle KMD$ uchburchak uchun kosinuslar teoremasini qo‘llab, quyidagi tenglikni topamiz:

$$KM^2 = KD^2 + DM^2 - 2 \cdot KD \cdot DM \cdot \cos(\angle B + \angle D)$$

Bu tenglikni yuqoridagi belgilashlar orqali yozsak,

$$n^2 = \frac{(ac)^2}{m^2} + \frac{(bd)^2}{m^2} - \frac{2abcd}{m^2} \cdot \cos(\angle B + \angle D)$$

ko‘rinishiga keladi. Tenglikni har ikkki tarafini m^2 ga ko‘paytirib,

$$(mn)^2 = (ac)^2 + (bd)^2 - 2abcd \cdot \cos(\angle B + \angle D)$$

tenglikni hosil qilamiz, ya’ni teorema to‘liq isbotlandi.▲

Izoh: Masalada isbotlangan tenglik Branshteyner teoremasi deyiladi. Undan quyidagi natijalar kelib chiqadi:

1 – Natija *(Ptolomey tengsizligi)* Ixtiyoriy $ABCD$ qavariq to‘rtburchak uchun ushbu tengsizlik $AB \cdot CD + AD \cdot BC \geq AC \cdot BD$ o‘rinli bo‘ladi.

Isbot: $\cos(\angle B + \angle D) \geq -1$ tengsizlikdan foydalanib quyidagi munosabatga ega bo‘lamiz:

$$(ac)^2 + (bd)^2 + 2abcd \geq (ac)^2 + (bd)^2 - 2abcd \cos(\angle B + \angle D) = (AC \cdot BD)^2$$

Ushbu tengsizlik **“Ptolomey tengsizligi”** deyiladi.

2 – Natija *(Ptolomey teoremasi)*. Qavariq $ABCD$ qavariq to‘rtburchak aylanaga ichki chizilgan bo‘lishi uchun $AB \cdot CD + AD \cdot BC = AC \cdot BD$ tenglik bajarilishi zarur va yetarli.

Isbot: 1-Natijaga ko‘ra $AB \cdot CD + AD \cdot BC \geq AC \cdot BD$ tengsizlikda tenglik holi faqat va faqat $\cos(\angle B + \angle D) = -1$ bo‘lgan holda ya’ni $\angle B + \angle D = \pi$ tenglik o‘rinli bo‘lganda bajariladi. Bu esa $ABCD$ qavariq to‘rtburchakka aylanaga ichki chizilgan bo‘lish shartidir▲

3 – masala. $x^3 - 4xy + y^3 = -1$ tenglamaning barcha butun yechimlarini toping.

Yechimi: Masalani yechish uchun quyidagi ayniyatdan foydalanamiz:

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

Masalada berilgan tenglamani 27 ga ko’paytirib,

$$(3x)^3 + (3y)^3 - 3 \cdot (3x)(3y) \cdot 4 + 4^3 = 4^3 - 1 = 63$$

ko'rinishida yozib olamiz. Demak

$$(3x+3y-4)(9x^2+9y^2+16+12x+12y-9xy)=63.$$

Tenglamaning o'ng tomoni 3 ga karrali, lekin chap tomoni 3 ga bo'linmaydi. U holda tenglama butun sonlarda yechimga ega emas ekan.

4 – masala. Aytaylik a, b, c, d, e, f natural sonlar uchun $ab+bc+ca-de-ef-fd$ va $abc+def$ ifodalar $S=a+b+c+d+e+f$ yig'indiga qoldiqsiz bo'linsin. U holda S ning murakkab son ekanligini isbotlang.

Yechim: Quyidagi ko'phadni qaraylik.

$$P(x)=(x+a)(x+b)(x+c)-(x-d)(x-e)(x-f).$$

Ko'rish mumkinki

$$P(x)=Sx^2+(ab+bc+ca-de-ef-fd)x+(abc+def)$$

va ko'phadning har bir koeffitsiyenti S ga qoldiqsiz bo'linadi, ya'ni ko'phadning ixtiyoriy butun sondagi qiymati ham S ga qoldiqsiz bo'linadi. Endi ko'phadning d dagi qiymatini hisoblaylik

$$P(d)=(a+d)(b+d)(c+d)$$

Agar S tub son bo'lsa, $a+d$, $b+d$, $c+d$ ko'paytuvchilarning birortasi S ga qoldiqsiz bo'linishi zarur. Lekin bu ziddiyat chunki, ularning har biri S dan kichik. Demak S murakkab son ekan.

5 – masala. $\triangle ABC$ uchburchakda ichida olingan har qanday M nuqta uchun

$$AB\cdot MA\cdot BM+AC\cdot MA\cdot CM+BM\cdot CM\cdot BC\geq AB\cdot AC\cdot BC$$

tengsizlik o'rinli bo'lishini isbotlang.

Isbot: $\triangle ABC$ uchburchak tekisligida quyidagi parallelogrammlarni yasaymiz:

$$\left.\begin{array}{r} ACBF\Rightarrow AC\parallel BF;\ FA\parallel BC \\ MCBE\Rightarrow MC\parallel BE;\ EM\parallel BC \end{array}\right\}\Rightarrow \left.\begin{array}{r} AF=EM=BC=a \\ BF=AC=b \end{array}\right\}$$

U holda $AFEM$ ham parallelogram, chunki $FA \parallel EM$ va $FA = EM$. To‘rtburchaklar uchun **Ptolomey tengsizligi**ni qo‘llab, quyidagi munosabatlarni topamiz:

$ABEF$ – to‘rtburchak uchun:

$$BE \cdot AF + AB \cdot FE \geq AE \cdot BF \Rightarrow a \cdot CM + c \cdot MA \geq b \cdot AE \quad \mathbf{(1)}$$

$AEBM$ – to‘rtburchak uchun:

$$BM \cdot AE + MA \cdot BE \geq EM \cdot AB \Rightarrow BM \cdot AE + MA \cdot CM \geq ac \quad \mathbf{(2)}$$

Endi ushbu tengsizliklardan foydalangan holda

$$cMA \cdot BM + aCM \cdot BM + bMA \cdot CM \geq BM\left(cMA + aCM\right) \overset{(1)}{\geq}$$

$$bBM \cdot AE + bMA \cdot CB = b\left(BM \cdot EA + MA \cdot CM\right) \overset{(2)}{\geq} abc$$

natijaga erishamiz. Masala to‘liq isbotlandi.

7-9 sinf o‘quvchilari uchun

1 – masala. Tenglamalar sistemasini yeching:

$$\begin{cases} a + b + c = 2020 \\ a^2 + b^2 + c^2 = 2020^2 \\ a^3 + b^3 + c^3 = 2020^3 \end{cases}$$

Yechim: Quyidagi ko‘phadni qaraylik

$$P(x) = (x-a)(x-b)(x-c).$$

Quyidagicha belgilash kiritaylik:

$$\begin{cases} a + b + c = S_1 \\ ab + bc + ca = S_2 \\ abc = S_3 \end{cases}.$$

Masalada berilgan shartlarga ko‘ra

$$S_1 = 2019, \; S_2 = \frac{S_1^{\,2} - (a^2 + b^2 + c^2)}{2} = 0$$

$$S_3 = \frac{(a^3+b^3+c^3) - S_1(S_1^{\,2} - 3S_2)}{3} = 0.$$

U holda

$$P(x) = (x-a)(x-b)(x-c) = x^3 - 2019x^2 = x^2(x-2019)$$

ekan. Demak (a,b,c) yechimlar $(2019,0,0)$ uchlikning o‘rin almashtirishlaridan iborat.

2 – masala. $x^9 + 5x^5 + x^4 + 4x + 4$ to‘rtta ko‘phadning ko‘paytmasi ko‘rinishida yozing.

Yechim: Dastlab quyidagi tenglikni hosil qilamiz:

$$x^9 + 5x^5 + x^4 + 4x + 4 = (x^9 + x^5 + x^4) + (4x^5 + 4x + 4) = (x^4 + 4)(x^5 + x + 1)$$

Shuningdek

$$x^4 + 4 = (x^2 + 2)^2 - 4x^2 = (x^2 - 2x + 2)(x^2 + 2x + 2)$$

va

$$x^5 + x + 1 = (x^3 - x^2 + 1)(x^2 + x + 1)$$

tengliklarni hisobga olsak,

$$x^9 + 5x^5 + x^4 + 4x + 4 = (x^2 + 2x + 2)(x^2 - 2x + 2)(x^3 - x^2 + 1)(x^2 + x + 1)$$

natijani topamiz.

3 – masala. Muntazam $\triangle ABC$ uchburchakka tashqi chizilgan aylananing B nuqtani o‘z ichiga olmagan AC yoyida olingan ixtiyoriy P nuqta uchun $BP = AP + PC$ tenglikni isbotlang.

Isboti: $\triangle$ Ma’lumki $ABCP$ - aylanaga ichki chizilgan to‘rtburchak. Unga **2 – natija**ni qo‘llab, ushbu $BP \cdot AC = AB \cdot CP + BC \cdot AP$ tenglikni hosil qilamiz. $AB = BC = CA$ ekanligini inobatga olsak, $BP = AP + CP$ tenglik hosil bo‘ladi.

4 – masala. Aytaylik $\triangle ABC$ uchburchakning balandliklari H nuqtada kesishsin. U holda

$$AB \cdot AH \cdot BH + AC \cdot CH \cdot AH + BC \cdot BH \cdot CH = AB \cdot BC \cdot AC$$

tenglik o‘rinli bo‘lishini isbotlang.

Isbot: Umuman olganda masalani

$$AH = 2R\cos\angle A,\ BH = 2R\cos\angle B,\ CH = 2R\cos\angle C$$

formulalar va hisoblashlar orqali isbotlash mumkin. Lekin biz boshqacharoq yo‘l tutamiz. 10-11 sinf o‘quvchilari uchun **5 – masala**da isbotlangan tengsizlikning tenglik holi bajarilishi uchun $ABEF$ va $AEBM$ to‘rtburchaklar aylanaga ichki chizilgan bo‘lishi zarur, ya’ni

$$\angle EBA + \angle EFA = \angle FEB + \angle FAB = 180^\circ$$

Demak $AMEF$ - to‘rtburchak ham aylanaga ichki chizilgan. Boshqa tomondan $AMEF$ parallelogram edi. U holda $FE \perp EM$ va $MA \perp BC$. Xuddi shunday $BM \perp AC$ va $\Rightarrow CM \perp AB$ ekanligi topishimiz mumkin. Demak M – balandliklar kesishgan nuqta bo‘lgan holda bajariladi.

5 – masala. O‘zaro turli va noldan farqli a, b, c haqiqiy sonlar uchun

$$\left(\frac{a^2}{bc}-1\right)^3+\left(\frac{b^2}{ca}-1\right)^3+\left(\frac{c^2}{ab}-1\right)^3=3\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}-\frac{bc}{a^2}-\frac{ca}{b^2}-\frac{ab}{c^2}\right)$$

tenglik o‘rinli. U holda $a+b+c$ ning qiymatini toping.

Yechish: Masalani ikkita usulini keltiramiz.

1 – usul: Agar $A=\frac{a^2}{bc}-1, B=\frac{b^2}{ca}-1, C=\frac{c^2}{ab}-1$ deb belgilash olsak,masalada berilgan ifoda

$$A^3+B^3+C^3=3ABC$$

tenglikka teng kuchli. Demak

$$A+B+C=0 \text{ yoki } A=B=C$$

Lekin ikkinchi holda $a=b=c$ kelib chiqadi va bu shartga zid. U holda

$$A+B+C=0 \Rightarrow a^3+b^3+c^3=3abc \Rightarrow a+b+c=0$$

2 – usul: $\frac{a^2}{bc}=x, \frac{b^2}{ca}=y, \frac{c^2}{ab}=z$ kabi belgilash olaylik. U holda masalada berilgan tenglik

$$(x-1)^3+(y-1)^3+(z-1)^3=3(x+y+z-xy-yz-zx)$$

ga teng kuchli. Ya'ni

$$x^3+y^3+z^3-3+3(x+y+z)-3(x^2+y^2+z^2)=3(x+y+z)-3(xy+yz+zx)$$

yoki $xyz=1$ ekanligini hisobga olsak,

$$x^3+y^3+z^3-3xyz=3(x^2+y^2+z^2-xy-yz-zx)$$

Endi $(*)$ ayniyatdan foydalanib oxirgi tenglik

$$(x+y+z-3)(x^2+y^2+z^2-xy-yz-zx)=0$$

ga teng kuchli. Demak

$$x+y+z=3 \Rightarrow a^3+b^3+c^3=3abc \Rightarrow a+b+c=0.$$

4-6 sinf o'quvchilari uchun

1. Raqam bilan yozing: ikki mlrd besh yuz o'n uch mln uch yuz ellik olti ming sakkiz yuz

A) 25133568 B) 25001330056 C) 2513356800 D) 20513035608

2. Qaysi eng katta

A) 1234567890 B) 987654321 C) 10203040506070809 D) 90807060504030201

3. Son nurida 423 sonidan oldin kelgan 3 ta sonni yozing.

A) 422, 423, 424 B) 421, 422, 423 C) 423, 424, 425 D) 420, 421, 422

4. 9 raqami ishtirok etgan barcha ikki xonali sonlar nechta?

A) 10 ta B) 19 ta C) 9 ta D) 18 ta

5. To‘g‘ri to‘rtburchakning eni 3 *m*, bo‘yi esa undan 2 *m* ortiq. Shu to‘g‘ri to‘rtburchakning perimetrini toping.

A) 8 *m* B) 16 *m* C) 5 *m* D) 15 *m*

6. To‘g‘ri chiziqing ikki nuqta bilan chegaralangan qismi qanday ataladi?

A) burchak B) nur C) uchburchak D) kesma

7. Kesma bir uchi tomonga cheksiz davom ettirilsa qanday shakl hosil bo‘ladi?

A) kesma B) nur C) to‘g‘ri chiziq D) tekislik

8. Ko‘paytmani toping. $548 \cdot 23$

A) 12508 B) 12604 C) 13501 D) 12406

9. 666 hosil qilish uchun qanday sonni 6 ga ko‘paytirish kerak?

A) 110 B) 112 C) 111 D) 110

10. Soat bilan ifodalang. 4 sutka

A) 96 B) 92 C) 94 D) 91

11. Kubning nechta qirralari bor.

A) 15 B) 13 C) 14 D) 12

12. Nodir 12 yoshda onasining yoshi esa undan 3 marta ortiq. Necha yildan so‘ng Nodir onasining hozirgi yoshida bo‘ladi?

A) 22 yildan so‘ng B) 23 yildan so‘ng C) 24 yildan so‘ng D) 25 yildan so‘ng

13. 3112 va 3424 sonlarini yig‘indisini 8 ga bo‘ling

A) 817 B) 781 C) 178 D) 165

14. Bo‘lishni bajaring. $26800 : 4$

A) 6700 B) 6800 C) 7100 D) 7200

15. To‘g‘ri to‘rtburchakning bo‘yi 45 *dm* eni 9 *dm* ga teng uning yuzi va perimetrini toping.

A) 408 dm^2 va 100 dm　　B) 301 dm^2 va 90 dm

C) 405 dm^2 va 108 dm　　D) 215 dm^2 va 54 dm

Test topshiriqlarining javoblari

1. C	**6. D**	**11. D**
2. D	**7. B**	**12. C**
3. D	**8. B**	**13. A**
4. D	**9. C**	**14. A**
5. B	**10. A**	**15. C**

Fan olimpiadalari bo‘yicha iqtidorli o‘quvchilar bilan ishlash departamenti sizga omadlar tilaydi!